Axioms of probability Lecture 0

Review of Infinite Series

Ex. Some of you have already taken a class with me and know that I'm a generous fellow. I want to give you a gift, which is the equivalent of countably many piles of money:

$$2^7 + 2^8 + 2^9 + 2^{10} + \ldots +$$

Do you want it?

Solution:

Assume that you will be getting something S

Then $$S = 2^7 + \underbrace{\left(2^8 + 2^9 + 2^{10} + \ldots +\right)}_{\text{Looks like } S}$$

$$= 2^7 + 2\underbrace{\left(2^7 + 2^8 + 2^9 + \ldots +\right)}_{S}$$

Hence $S = 2^7 + 2S$ or $S = -2^7 = -128$.

You owe me \$128! And FYI Das Gift means poison in German.

Some of you may object that ∞ is also a solution to the equation

$$S = 2^7 + 2S$$

However, strictly speaking ∞ is a symbol that represents growth without bound. It is not a number. Not a "definite something". Moreover, when solving physics problems you dismiss infinite solutions as invalid.

You might have learned in calculus that the series diverges, but see notes on infinite limits from Calculus 1.

For applications in probability theory, we will use the standard definition of series, which we will review below.

Sequences

A sequence is a function $f: \mathbb{N} \longrightarrow X$ whose domain is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$
If $f(n) = x$, we typically write x_n instead of $f(n)$.

Ex. (a) $f(n) = \frac{1}{2n+1}$ is the sequence

$\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots, \}$

(b) The sequence $\{1, 2, 6, 24, 120, \dots\}$

can be writen as $a_n = n!$

We can write the sequence a_n in a column. If the rows eventually look the same to any fixed observer we say that the sequence converges. For example, suppose the sequence is $a_n = 10^{-n}$ and that you can only resolve the first 5 decimal points (your $\varepsilon = 10^{-6}$) then

n	a_n
1	$\boxed{0.10000}00\ldots$
2	$\boxed{0.01000}00\ldots$
3	$\boxed{0.00100}000\ldots$
4	$\boxed{0.00010}000\ldots$
5	$\boxed{0.00001}000\ldots$
6	$\boxed{0.00000}100\ldots$
7	$\boxed{0.00000}0100\ldots$

The sequence looks like 0 from the 7^{th} row onwards

This is the idea behind the following definition.

Def: A sequence a_n converges to L if for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that $|a_n - L| < \varepsilon$ whenever $n \geq N(\varepsilon)$

We express this as $\lim_{n\to\infty} a_n = L$.

Remark: The definition means that to a detector of resolution $\varepsilon > 0$ the sequence a_n is indistinguishable from the number L for any row $n \geq N(\varepsilon)$ (starting from $N(\varepsilon)$ onwards).

You can find a review of δ-ε arguments in my calculus notes.

Important limits

What is $\lim_{h \to 0} (1+h)^{1/h}$, do you recall?

This limit has a lot to do with solving the differential equation $\frac{d}{dx} f(x) = f(x)$.

Recall that if $f(x) = a^x$ then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h \to 0} \frac{a^{0+h} - a^0}{h}$$

$$= a^x f'(0) = c\,a^x = c\,f(x) \text{ if } \lim_{h \to 0} \frac{a^h - 1}{h}$$

exists.

Thus, we make the inspired guess that there exists a number $e > 0$ such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. We will say more about this number later.

This implies that $\frac{d}{dx} e^x = e^x$.

Observe that if $f'(x) = f(x)$ then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f(x)$$

or
$$\frac{f(x+h) - f(x)}{h} \approx f(x)$$

for very small h.

Thus $\quad f(\underbrace{x+h}_{u}) \approx (1+h) f(\underbrace{x}_{u-h})$.

This means that $f(\#) \overset{\text{ess.}}{=} (1+h) f(\#-h)$

or evaluating at a number is essentially the same as $\left(1 + \text{small step}\right) \cdot$ function evaluated at a number one step prior. It gives us a recursive formula.

$f(x) \approx (1+h) f(x-h)$

Setting $f(x) = e^x$ we see that $f(0) = e^0 = 1$.

Divide the interval $[0, x]$ into n subintervals, where n is very large. Then set $h = \frac{x}{n}$.

$$f(x) \approx \left(1+\frac{x}{n}\right) f\left(x-\frac{x}{n}\right) \approx \left(1+\frac{x}{n}\right)^2 f\left(x-2\frac{x}{n}\right)$$

$$\approx \left(1+\frac{x}{n}\right)^3 f\left(x-3\frac{x}{n}\right) \approx \dots \left(1+\frac{x}{n}\right)^n f\left(x-n\frac{x}{n}\right)$$

$$= \left(1+\frac{x}{n}\right)^n f(0) = \left(1+\frac{x}{n}\right)^n$$

The approximations become better and better if we push n to ∞.

Hence

$$e^x = f(x) = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n.$$

<u>Ex.</u> Calculate

(a) $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n$

(b) $\lim\limits_{n\to\infty} \left(1-\frac{5}{n}\right)^n$

(c) $\lim\limits_{h\to 0} (1+h)^{1/h}$

(d) $\lim\limits_{h\to 0} (1+5h)^{1/h}$

(e) $\lim\limits_{h\to 0} (1-2h)^{1/h}$

(f) $\lim\limits_{h\to 0} (1+xh)^{1/h}$

Solution:

(a) $\lim\limits_{n \to \infty} \left(1 + \frac{\boxed{1}}{n}\right)^n = e^{\boxed{1}}$

(b) $\lim\limits_{n \to \infty} \left(1 + \frac{\boxed{-5}}{n}\right)^n = e^{\boxed{-5}}$

(c) Note the pattern $(1 + \square)^{\frac{1}{\square}} \approx e$ if $\square \approx 0$.

This can be established from $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ or we could think as follows

$\frac{d}{dx} \ln x = \frac{1}{x}$ so $\frac{d}{dx} \ln x \Big|_{x=1} = \frac{1}{1} = 1$

By definition of derivative $\frac{d}{dx} \ln x \Big|_{x=1} =$

$= \lim\limits_{h \to 0} \frac{\ln(1+h) - \ln(1)}{h} = \lim\limits_{h \to 0} \frac{1}{h} \ln(1+h)$

$$= \lim_{h \to 0} \ln (1+h)^{1/h} = 1$$

Hence $\quad e^1 = e^{\lim_{h \to 0} (1+h)^{1/h}} = \lim_{h \to 0} e^{\ln(1+h)^{1/h}}$

$$= \lim_{h \to 0} (1+h)^{1/h}.$$

(d) $\displaystyle \lim_{h \to 0} \left(1+\boxed{5h}\right)^{\frac{1}{\boxed{h}}} = \lim_{h \to 0} \left(1+\boxed{5h}\right)^{\frac{1}{\boxed{5h}} \cdot 5}$

$= e^5$

(e) $\displaystyle \lim_{h \to 0} (1-2h)^{1/h} = \lim_{h \to 0} \left(1+\boxed{-2h}\right)^{\frac{1}{\boxed{-2h}} (-2)}$

$= e^{-2}.$

(f) $\lim_{h \to 0} (1+xh)^{\frac{1}{h}} = \lim_{h \to 0} (1+\boxed{xh})^{\frac{1}{\boxed{xh}} \cdot x}$

Series

Given a sequence $\{a_1, a_2, a_3, \dots\}$ we can try to sum it up:

$$a_1 + a_2 + a_3 + a_4 + \dots + = \sum_{k=1}^{\infty} a_k$$

The meaning of this sum is taken to be the limit of partial sums:

$$S_1 = a_1 = \sum_{k=1}^{1} a_k$$

$$S_2 = a_1 + a_2 = \sum_{k=1}^{2} a_k$$

$$S_3 = a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k$$

$$\vdots$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^{n} a_k$$

Thus the series is viewed as just another sequence. If $\lim_{n\to\infty} S_n = S$, we will see the rows become more and more the same.

Specifically, for any fixed detector with resolution $\varepsilon > 0$, this detector will conclude that $S_n = S$ starting from row $N(\varepsilon)$

Here is a simple consequence:

<u>Thm:</u> If $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = S$ then $\lim_{n\to\infty} a_n = 0$.

<u>Proof:</u> Set $S_n = \sum_{k=1}^{n} a_k$. Then

$$a_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0$$

Remark: S_{n-1} laggs one row behind S_n in convergence to S.

Ex. The sum $2^7 + 2^8 + 2^9 + \ldots +$ is interpreted as

1. 2^7
2. $2^7 + 2^8$
3. $2^7 + 2^8 + 2^9$
4. $2^7 + 2^8 + 2^9 + 2^{10}$

$\vdots$

Clearly $S_n = \sum_{k=1}^{n} 2^{6+k}$ diverges to ∞.

This is not the same as giving the sum "all at once".
See calculus 1 notes on limits at infinity.

Geometric Series

The simplest series perhaps is of the form

$$ar^m + ar^{m+1} + ar^{m+2} + \ldots + = \sum_{k=1}^{\infty} ar^{m-1+k}$$

It is easy to see that it converges if and only if

$-1 < r < 1$.

The sum of a geometric series is very easy to find

$$S = ar^m + \underbrace{\left(ar^{m+1} + ar^{m+2} + ar^{m+3} + \dots +\right)}_{\text{Almost } S}$$

$$= ar^m + r\underbrace{\left(ar^m + ar^{m+1} + ar^{m+2} + ar^{m+3} + \dots +\right)}_{S}$$

$$= ar^m + rS$$

Hence $(1-r)S = ar^m$.

or $$S = \frac{ar^m}{1-r}$$

see lecture notes for calculus 1 (on limits at infinity) for more examples.

Taylor Series

The simplest operations and perhaps the only operations that we know how to implement are $\times, +$. $\sqrt[n]{\cdot}$, $\int_a^b \cdot$ etc are operations that we describe using $+$ and $\times$.

A function $p(x)$ that only employs addition and multiplication is called polynomial.
We can group together operations by the number of times we multiplied by the input.

$$p(x) = \underbrace{a_0}_{\text{input not used}} + \underbrace{a_1 x}_{\text{input used once}} + \underbrace{a_2 x^2}_{\text{input used twice}} + \dots + \underbrace{a_n x^n}_{\text{input used } n \text{ times.}}$$

Ex. Is $\sqrt{x}$ a polynomial?

Solution:

We are asking whether or not the square root operation can be accomplished using multiplication and addition alone.

Notice that $\sqrt{x}$ is not defined for negative numbers, whereas polynomial operations are defined for all numbers. Thus $\sqrt{x}$ cannot be modeled by operations which involve finitely many additions and multiplications.

What happens when we allow infinitely many additions and multiplications to be carried out?

Def: A power series is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + \; = \sum_{k=0}^{\infty} a_k x^k$$

It is merely an infinite polynomial.

Problem: Suppose we want to represent $f(x) = e^x$ as a power series. How do we do that?

Solution:

Suppose $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots +$

What are the coefficients $a_0, a_1, a_2 \ldots$?

Observe that plugging 0 for x, we eliminate every coefficient save for a_0:

$$\boxed{f(0) = a_0}$$

We can get a_1 isolated by taking the derivative

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots +$$

Plugging 0, we get

$$\boxed{f'(0) = a_1}$$

In general, to find a_k, take the k^{th} derivative

$$f^{(k)}(x) = k!\, a_k + (k+1)(k)\cdots(2)\, a_{k+1} x + \ldots +$$

Plugging 0, we get

$$\boxed{f^{(k)}(0) = k!\, a_k}$$

Thus $a_k = \dfrac{f^{(k)}(0)}{k!}$

We see that

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k}$$

This is called Taylor's form. Since

$\frac{d^k}{dx^k}(e^x) = e^x$

We see that $f^{(k)}(0) = f(0) = e^0 = 1$.

Hence

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Ex. (a) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} = e^1$ (b) $\displaystyle\sum_{k=0}^{\infty} \frac{5^k}{k!} = e^5$

(c) $\displaystyle\sum_{k=0}^{\infty} \frac{(-3)^k}{k!} = e^{-3}$

__Remark:__ We can construct a solution to the equation

$$\frac{d}{dx} f(x) = f(x) \;;$$

$$f(0) = 1$$

Simply in terms of power series alone!

The function $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ can be shown to be the unique solution to this equation.

Moreover $f(x)$ can be shown to have properties that we ascribe to exponential functions.

Can you figure out how to do this?